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COMMENT

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Abstract. The pointwise convergence of the probability densities induced from an initial density on (0, 1) under repeated iteration of the map $x \mapsto 4x(1-x)$ is established for a large class of initial densities.

Let f_0 be a probability density function on [0, 1], and let f_n denote the density on [0, 1] induced from f_0 by n applications of the map $x \mapsto h(x) := 4x(1-x)$. In Grosjean (1986) it is shown, using the results of Falk (1984) and Nandakumaran (1985), that $f_n(x) \to \{\pi[x(1-x)]^{1/2}\}^{-1}$ for all $x \in (0, 1)$, provided that the symmetric part f_0^{sym} of f_0 with respect to the point $x = \frac{1}{2}$ belongs to the class $\mathcal{T}$ of functions having an expansion of the form $a_0 + \Sigma_{n\geq 1} a_n T_n[1-8x(1-x)]$, in terms of the Chebyshev polynomials T_n of the first kind, for which $\Sigma_{n\geq 0}|a_n| < \infty$.

This implies, in particular, that f_0^{sym} must be continuous, but Grosjean conjectures that f_n converges even if f_0^{sym} has a number of jumps. It should also be noted that the class $\mathcal{T}$ does not contain the invariant density $\{\pi[x(1-x)]^{1/2}\}^{-1}$. In this comment, a very simple argument is used to prove that f_n converges for a class of initial densities which includes the invariant density and those conjectured by Grosjean.

Theorem. Suppose that the function $g_0 : x \mapsto [x(1-x)]^{1/2} f_0^{\text{sym}}(x)$ belongs to the space $D[0, 1]$ of functions on [0, 1] which are right continuous and have left limits. Then

$$\lim_{n\to\infty} \sup_{0<x<1} |\pi[x(1-x)]^{1/2} f_n(x) - 1| = 0.$$

In particular,

$$\lim_{n\to\infty} f_n(x) = \{\pi[x(1-x)]^{1/2}\}^{-1}$$

for all $x \in (0, 1)$.

Proof. Let $g_n(x) := [x(1-x)]^{1/2} f_n(x)$ for $n \geq 1$. Since the functions f_n obey the recursion

$$f_{n+1}(x) = \tfrac{1}{4}(1-x)^{-1/2}(f_n\{\tfrac{1}{2}[1+(1-x)^{1/2}]\} + f_n\{\tfrac{1}{2}[1-(1-x)^{1/2}]\}) \qquad x \in (0, 1) \qquad (1)$$

it follows that

$$g_{n+1}(x) = \tfrac{1}{2}(g_n\{\tfrac{1}{2}[1+(1-x)^{1/2}]\} + g_n\{\tfrac{1}{2}[1-(1-x)^{1/2}]\}) \qquad x \in (0, 1). \qquad (2)$$

Let g_n be a step function of the form

$$g_n(x) = a_n I[x < c_n] + u_n I[x = c_n] + b_n I[x > c_n] \qquad (3)$$

where $u_n \in \{a_n, b_n\}$. Then g_{n+1} is also a step function of the form (3), satisfying

$$\max(a_n, b_n) \geq \max(a_{n+1}, b_{n+1}) \geq \min(a_{n+1}, b_{n+1}) \geq \min(a_n, b_n)$$

and

$$|a_{n+1} - b_{n+1}| \leq \tfrac{1}{2}|a_n - b_n|.$$

For example, taking $c_n > \frac{1}{2}$,

$$g_{n+1}(x) = \tfrac{1}{2}(a_n + b_n)I[x < c_{n+1}] + \tfrac{1}{2}(a_n + u_n)I[x = c_{n+1}] + a_n I[x > c_{n+1}]$$

where $c_{n+1} = 1 - (2c_n - 1)^2$. Hence it follows that, if g_0 is of the form (3), $g_n(x)$ converges uniformly in $x \in (0, 1)$ to the constant $g_\infty := \lim_{n\to\infty} a_n$. Note that

$$g_\infty = \int_0^1 \frac{g_0(x)}{\pi[x(1-x)]^{1/2}} \,\mathrm{d}x$$

since, from (1), $\int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 f_0(x)\,\mathrm{d}x$ for all n.

Since also, from (2),

$$\sup_{0<x<1} |g_{n+1}(x)| \leq \sup_{0<x<1} |g_n(x)|$$

it follows that g_n converges uniformly to $\int_0^1 g_0(x)\{\pi[x(1-x)]^{1/2}\}^{-1}\,\mathrm{d}x$ whenever g_0 can be uniformly approximated by finite linear combinations of simple step functions, a class of functions including $D[0, 1]$ (Billingsley 1968, ch 3, lemma 1). Of course, if f_0 is a probability density on (0, 1), $\int_0^1 g_0(x)\{\pi[x(1-x)]^{1/2}\}^{-1}\,\mathrm{d}x = \pi^{-1}$.

An analogous result may also be established for many piecewise expansive Markov maps, using more complicated arguments, similar to those in Collet and Eckmann (1985), § II. The special case of $h(x) := 1 - 2|x - \frac{1}{2}|$ can then be transformed into $\tilde{h}(x) := 4x(1-x)$ in the usual way.

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